

# Research Note on New Similarity Reductions of a Variable-Coefficient Korteweg-de Vries Equation

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Z. Naturforsch. **52a**, 463–464 (1997);  
received February 7, 1997

For a variable-coefficient Korteweg-de Vries equation we obtain 4 new similarity reductions to the Painlevé type equations or the Weierstrass elliptic function equation.

In this note, we report the application of the Clarkson-Kruskal direct method [1], which is powerful for the constant-coefficient nonlinear evolution equations, to a variable-coefficient Korteweg-de Vries equation [2],

$$u_t + f(t) u u_x + g(t) u_{xxx} = 0, \quad (1)$$

where  $f(t) \neq 0$  and  $g(t) \neq 0$ .

Firstly, it is sufficient to seek a similarity reduction of (1) in the form

$$u(x, t) = a(x, t) + b(x, t) w[z(x, t)], \quad (2)$$

where the functions  $a(x, t)$ ,  $b(x, t)$ ,  $z(x, t)$  and  $w(z)$  are to be determined, rather than the more general form of  $u(x, t) = U\{x, t, w[z(x, t)]\}$  (proof ignored). Also, we will consider the interesting case of  $z_x \neq 0$ .

After the substitution of (2), we impose the condition that (1) be an ordinary differential equation for  $w(z)$ , i.e., demand the ratios of the coefficients of different derivatives and powers of  $w(z)$  to be functions of  $z$  only. Computerized symbolic computation thus leads to a set of conditions for  $a(x, t)$ ,  $b(x, t)$  and  $z(x, t)$ , based on which we find the following format of similarity reductions:

$$u(x, t) = -\frac{\theta'(t) x + \phi'(t)}{f(t) \theta(t)} + \frac{g(t)}{f(t)} z_x^2(x, t) w[z(x, t)], \quad (3)$$

$$\text{with } z(x, t) = \theta(t) x + \phi(t), \quad (4)$$

where the functions  $\theta(t) \neq 0$ ,  $\phi(t)$ ,  $w(z)$ , as well as the constraints on  $f(t)$  and  $g(t)$ , are given below case by case:

Similarity reduction I:

$$\theta(t) = k, \quad (5)$$

$$\phi(t) = \begin{cases} C k^6 \int^t f(s) \int^s \frac{g^2(s)}{f(s)} ds dt & \text{when } C \neq 0, \\ M \int^t f(s) ds & \text{otherwise,} \end{cases} \quad (6)$$

where  $C$ ,  $k \neq 0$  and  $M$  are arbitrary constants, while  $w(z)$  satisfies the ordinary differential equation

$$w''' + w w' + C = 0, \quad (7)$$

which is equivalent to either the first Painlevé equation as  $C \neq 0$  or the Weierstrass elliptic function equation as  $C = 0$ .

The constraint on  $f(t)$  and  $g(t)$  is

$$A \left[ \frac{g(t)}{f(t)} \right] = 0, \quad \text{where } A \equiv \frac{d}{dt} \ln. \quad (8)$$

Similarity reduction II:

$$\theta(t) = E \frac{f(t)}{g(t)}, \quad (9)$$

while  $\phi(t)$  satisfies

$$\frac{f(t)}{g^2(t) \theta^5(t)} \left[ \frac{f'(t) \phi'(t)}{f^2(t) \theta(t)} + \frac{2 \theta'(t) \phi'(t)}{f(t) \theta^2(t)} \frac{\phi''(t)}{f(t) \theta(t)} \right] = B \phi(t) + D, \quad (10)$$

where  $D$  and  $E \neq 0$  are arbitrary constants, while  $B$  is a constant given by (13) below, and  $\theta(t)$  by (9).  $w(z)$  must satisfy

$$w''' + w w' + B z + D = 0, \quad (11)$$

which is equivalent to the first Painlevé equation when  $B = 0$ , or else, not of the Painlevé type.

The constraints on  $f(t)$  and  $g(t)$  are

$$A \left[ \frac{g(t)}{f(t)} \right] \neq 0, \quad (12)$$

$$\frac{f(t)}{g^2(t) \theta^6(t)} \left\{ \frac{f'(t) \theta'(t)}{f^2(t) \theta(t)} + \frac{2 [\theta'(t)]^2}{f(t) \theta^2(t)} - \frac{\theta''(t)}{f(t) \theta(t)} \right\} = B = \text{constant}, \quad (13)$$

with  $\theta(t)$  given by (9).

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Similarity reduction III:

$$\theta(t) = -[3A \int g(t) dt]^{-1/3}, \quad (14)$$

$$\phi(t) = \frac{H}{2A^2} - [3A \int g(t) dt]^{-1/3}, \quad (15)$$

where  $A \neq 0$  and  $H$  are arbitrary constants, while  $w(z)$  satisfies

$$w''' + w w' + A w - 2A^2 z + H = 0, \quad (16)$$

the solutions of which are known to be related through a one-to-one transformation to those of the second Painlevé equation.

The constraint on  $f(t)$  and  $g(t)$  is

$$A \left[ \frac{g(t)}{f(t)} \right] = 0, \quad \text{where } A \equiv \frac{d}{dt} \ln. \quad (17)$$

Similarity reduction IV:

$$\theta(t) = -\frac{f(t)}{g(t)} \left[ 3N \int \frac{f^3(t)}{g^2(t)} dt \right]^{-1/3}, \quad (18)$$

$$\phi(t) \begin{cases} \text{satisfies} & \frac{f'(t)\phi'(t)}{f^2(t)\theta(t)} + \frac{2\theta'(t)\phi'(t)}{f(t)\theta^2(t)} - \frac{\phi''(t)}{f(t)\theta(t)} = P \frac{g^2(t)}{f(t)} \theta^5(t) & \text{if } P \neq 0 \text{ \& } R=0, \\ = \theta(t) & & \text{if } P=0 \text{ \& } R=0, \\ = \theta(t) - P/R & & \text{otherwise,} \end{cases} \quad (19)$$

where  $N \neq 0$  and  $P$  are arbitrary constants, while  $R$  is a constant given by (22) below, and  $\theta(t)$  by (18).  $w(z)$  satisfies

$$w''' + w w' + N w + R z + P = 0, \quad (20)$$

which is a form of the second Painlevé equation only when  $R + 2N^2 = 0$ .

The constraints on  $f(t)$  and  $g(t)$  are

$$A \left[ \frac{g(t)}{f(t)} \right] \neq 0, \quad (21)$$

$$\frac{R}{N^2} = -2 + 9 \frac{g^4(t)}{f^5(t)} \left[ \int \frac{f^3(t)}{g^2(t)} dt \right]^2 \cdot \left\{ \frac{3[f'(t)]^2}{f^3(t)} - \frac{3f'(t)g'(t)}{f^2(t)g(t)} - \frac{f''(t)}{f^2(t)} + \frac{g''(t)}{f(t)g(t)} \right\}, \quad (22)$$

with  $\theta(t)$  given by (18).

*Acknowledgement*

This work has been supported by the Outstanding Young Faculty Fellowship and the Research Grants for the Scholars Returning from Abroad, State Education Commission of China.

[1] P. Clarkson and M. Kruskal, J. Math. Phys. **30**, 2201, 1989.

[2] N. Joshi, Phys. Lett. **A 125**, 456, 1987.